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Propagation of Chaos and the Hopf–Cole Transformation

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DEDICATED TO THE MEMORY OF MARK KAC

It was shown in [2] that Burgers' equation can be obtained as a contraction of a certain N-body problem when $N \to \infty$. Using this result we derive a variant of the Hopf–Cole transformation.

0. INTRODUCTION

The Hopf–Cole transformation [3, 1]

$$v = e^{\int_{-\infty}^{x} u} \tag{0.1}$$

carries solutions of the Burgers equation

$$u_t = u_{xx} + 2uu_x \tag{0.2}$$

into solutions of the heat equation $v_t = v_{xx}$. The author and M. Kac have shown in [2] that Burgers' equation is the limit when $N \to \infty$ of a contracted N-body problem

$$\frac{\partial F}{\partial t} = \sum_{i=1}^{N} \frac{\partial^2 F}{\partial x_i^2} + \sum_{1 \le i < j \le N} \delta(x_i - x_j)\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right) F. \tag{0.3}$$

The operator $L = \sum_{i=1}^{N} \partial^2/\partial x_i^2 + \sum_{i<j} \delta(x_i - x_j)(\partial/\partial x_i + \partial/\partial x_j)$ can be transformed into the Laplacean $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$ by an intertwining operator Q [2]. Using this connection we have attempted in [2] to obtain a linearizing transformation for the Burgers equation hoping to get the Hopf–Cole transformation. It is clear now that we missed a turn and got lost in horrendous calculations. In this paper, I overcome (or rather avoid) the technical difficulties of [2] and obtain the transformation

$$v = \frac{\partial}{\partial x} e^{\frac{1}{2}[\int_{-\infty}^{x} u - \int_{x}^{\infty} u]} \tag{0.4}$$

which is a close relative of the Hopf–Cole transformation.

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It would be interesting to understand which nonlinear evolution equations can be obtained by contraction from linear N-body problems. It is even more interesting to distinguish those nonlinear equations that can be linearized using this approach. I hope to return to these questions in a future publication.

The problem was suggested by Mark Kac and I take this opportunity to thank him for his cheerful encouragement.

1. The N-Body Problem

Here I briefly summarize the results of [2]. It is convenient to introduce a coupling constant c into the Burgers equation

$$u_t = u_{xx} + 2cuu_x. \tag{1.1}$$

For $N = 2, 3, \dots,$ set

$$L_N = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{c}{N} \sum_{1 \leq i \leq j \leq N} \delta(x_i - x_j)\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right) \tag{1.2}$$

where $\delta(x)$ is the Dirac's delta function.
The evolution equation

$$\frac{\partial}{\partial t} F = L_N F \tag{1.3}$$

with initial condition $F(0; x_1, \dots, x_N)$ corresponds to the diffusion of N coupled Brownian particles on the line [4]. Operator L_N is symmetric with respect to permutations of $x_1, \dots, x_N$ and we restrict L_N to the space of symmetric functions of N variables. Denote by $\mathbb{R}_0^N$ the domain in $\mathbb{R}^N$ given by inequalities $x_1 \leq x_2 \leq \dots \leq x_N$. The symmetric part of L_N is equivalent to the Laplacean $\Delta_N = \sum_{i=1}^N \partial^2/\partial x_i^2$ in $\mathbb{R}_0^N$ with the boundary conditions

$$\left(\frac{\partial}{\partial x_{k+1}} - \frac{\partial}{\partial x_k}\right)F + \frac{c}{2N}\left(\frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_{k+1}}\right)F = 0 \qquad \text{for } k = 1, \dots, N-1. \tag{1.4}$$

From now on we interpret the Eq. (1.3) as an initial value problem in $\mathbb{R}_0^N$ with the boundary conditions (1.4). In [2] we exhibit an operator Q_N that intertwines the symmetric Laplacean Δ_N with L_N. We recall the construction of Q_N here. For any $i, j = 1, \ldots, N$ define the operator $Q_{i,j}(c)$ by

$$\begin{aligned}(Q_{i,j}(c)F)(x_1,\ldots,x_N) = &\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right)\\ &\times \int_0^\infty dt\, F\biggl(x_1,\ldots,x_i + \left(\frac{c}{2N} - 1\right)t,\ldots,x_j\\ &\qquad + \left(\frac{c}{2N} + 1\right)t,\ldots,x_N\biggr).\end{aligned} \tag{1.5}$$

Let S be the operator from all functions F on $\mathbb{R}^N$ into symmetric functions on $\mathbb{R}^N$ obtained by restricting F to $\mathbb{R}_0^N$ and then extending it to a symmetric function on $\mathbb{R}^N$. The formula for Q_N can be written [2] as

$$Q_N = S\prod_{i<j}\left(1 + \frac{c}{2N}Q_{i,j}(c)\right).[1] \tag{1.6}$$

It is shown in [2] that Q_N has the intertwining property

$$L_N Q_N = Q_N \Delta_N. \tag{1.7}$$

Equation (1.7) means that Q_N takes Laplacean in $\mathbb{R}_0^N$ with Neumann boundary conditions

$$\left(\frac{\partial}{\partial x_{k+1}} - \frac{\partial}{\partial x_k}\right)F = 0, \qquad k = 1,\ldots,N-1 \tag{1.8}$$

into the Laplacean with the "oblique derivative" boundary conditions (1.4).

2. Contraction of the N-Body Problem

Consider the initial value problem (1.3) with the initial condition

$$F_N(0; x_1,\ldots,x_N) = f_0(x_1)\cdots f_0(x_N) \tag{2.1}$$

where f_0 is a fixed smooth function normalized by

$$\int_{-\infty}^{\infty} f_0(x)\,dx = 1. \tag{2.2}$$

For $1 \le n \le N$ denote by $F_{n,N}$ the contracted function

$$F_{n,N}(t; x_1,\ldots,x_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_{n+1}\cdots dx_N F_N(t; x_1,\ldots,x_N). \tag{2.3}$$

[1] Intertwining operators similar to Q_N are useful in the study of the delta Bose gas and the quantum nonlinear Schrödinger equation (cf. [5]).

It was shown in [2] that the limit $f_n = \lim_{N \to \infty} F_{n,N}$ exists for $n = 1, 2, \ldots$ and satisfies the "propagation of chaos" property:

$$f_n(t; x_1, \ldots, x_n) = f_1(t; x_1) \cdots f_1(t; x_n). \tag{2.4}$$

In what follows we denote the function $f_1(t; x)$ by f. Then $f(t; x)$ satisfies the Burgers Eq. (1.1) with the initial condition $f(0; x) = f_0(x)$ [2].

3. Linearizing Transformation

Using the results of Sections 1 and 2 we obtain an explicit transformation linearizing (1.1). Equation (1.7) implies

$$L_N = Q_N \Delta_N Q_N^{-1}. \tag{3.1}$$

Exponentiating both sides of (3.1) we get

$$e^{tL_N} = Q_N e^{t\Delta_N} Q_N^{-1} \tag{3.2}$$

which implies

$$e^{tL_N} Q_N = Q_N e^{t\Delta_N}. \tag{3.3}$$

We take the adjoint to (3.3) with respect to the usual inner products in $L_2(\mathbb{R}^N)_{\mathrm{sym}} \subset L_2(\mathbb{R}^N)$:

$$Q_N^* e^{tL_N^*} = e^{t\Delta_N} Q_N^*. \tag{3.4}$$

Taking the adjoint changes c to $-c$ in the expression (1.2) for L_N. Thus we have $L_N(c)^* = L_N(-c)$ in an obvious notation. Using the same notation we set $R_N = Q_N(-c)^*$ and rewrite (3.4) as

$$R_N e^{tL_N} = e^{t\Delta_N} R_N. \tag{3.5}$$

For any $F_N \in L_2(\mathbb{R}^N)_{\mathrm{sym}}$ the vector $F_N(t) = e^{tL_N} F_N$ is the solution of the initial value problem (1.3) with the initial condition F_N and we have according to (3.5)

$$R_N e^{tL_N} F_N = e^{t\Delta_N} R_N F_N. \tag{3.6}$$

Denote by $C_{n,N} (1 \le n \le N)$ the operator of contracting variables $x_{n+1}, \ldots, x_N$; i.e.,

$$(C_{n,N} F)(x_1, \ldots, x_N) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_{n+1} \cdots dx_N F(x_1, \ldots, x_N). \tag{3.7}$$

Since the right-hand side of (3.6) satisfies the heat equation in N dimensions the function $\varphi_N(t; x_1) = C_{1,N} e^{t\Delta_N} R_N F_N$ satisfies the heat equation in variables t and x_1 with the initial condition $C_{1,N}R_NF_N$. By (3.6)

$$\varphi_N = C_{1,N}R_N e^{tL_N}F_N \tag{3.8}$$

so φ_N is expressed in terms of the solution of (1.3) with initial condition F_N. It remains to show that when F_N is given by (2.1), $\lim_{N\to\infty}\varphi_N = \varphi$ exists and obtain an expression for φ in terms of $f = \lim_{N\to\infty}C_{1,N}e^{tL_N}F_N$. We do this in the rest of the paper.

4. Explicit Computation

Denote by $h(x)$ the characteristic function of $[0, \infty)$ (Heavyside function). Expanding (1.6) in powers of c and rearranging terms in the expansion we get

$$Q_N = \sum_{n=0}^{N(N-1)/2}\left(\frac{c}{2N}\right)^n \sum_{(i_1,j_1),\ldots,(i_n,j_n)} h(x_{j_1}-x_{i_1}) \cdots h(x_{j_n}-x_{i_n})Q_{i_1,j_1}(c)\cdots Q_{i_n,j_n}(c). \tag{4.1}$$

The second summation in (4.1) is over all sets of n ordered pairs of indices. Notice that operators $Q_{i,j}(c)$ all commute so the order of pairs $(i_1, j_1),\ldots,(i_n, j_n)$ does not matter. An elementary computation shows that $Q_{i,j}(c)^* = -Q_{j,i}(-c)$. Thus (4.1) implies

$$R_N = \sum_{n=0}^{N(N-1)/2}\left(\frac{c}{2N}\right)^n \sum_{(i_1,j_1),\ldots,(i_n,j_n)} Q_{i_1,j_1}(c) \cdots Q_{i_n,j_n}(c)h(x_{i_1}-x_{j_1})\cdots h(x_{i_n}-x_{j_n}). \tag{4.2}$$

The operator $Q_{i_1,j_1}(c)\cdots Q_{i_n,j_n}(c)$ has factor $(\partial/\partial x_{i_1} + \partial/\partial x_{j_1})\cdots(\partial/\partial x_{i_n} + \partial/\partial x_{j_n})$ on the left. So $C_{1,N}Q_{i_1,j_1}(c)\cdots Q_{i_n,j_n}(c)$ is zero unless all pairs $(i_1, j_1),\ldots,(i_n, j_n)$ contain index 1. If (i, j) and (j, i) both occur among pairs $(i_1, j_1),\ldots,(i_n, j_n)$ the corresponding term in (4.2) will be zero because $h(x_i - x_j)h(x_j - x_i) = 0$. Thus we have

$$C_{1,N}R_N = \sum_{n=0}^{N(N-1)/2}(c/2N)^n \sum_{\substack{1<i_1<\cdots<i_p\le N\\ 1<j_1<\cdots<j_q\le N\\ p+q=n}} C_{1,N}Q_{1,i_1}(c)\cdots Q_{1,i_p}(c)Q_{j_1,1}(c)\cdots Q_{j_q,1}(c)h(x_1-x_{i_1})\cdots h(x_1-x_{i_p})h(x_{j_1}-x_1)\cdots h(x_{j_q}-x_1). \tag{4.3}$$

When $C_{1,N}R_N$ is applied to a symmetric function Φ_N of $x_1, \dots, x_N$ the terms corresponding to multiindices $(i_1, \dots, i_p; j_1, \dots, j_q)$ and $(i'_1, \dots, i'_p; j'_1, \dots, j'_q)$ in (4.3) are equal if one multiindex is obtained from another by a permutation of $(2, \dots, N)$. Thus we can rewrite (4.3) as

$$C_{1,N}R_N\Phi_N = \sum_{\substack{p,q\geq 0 \\ p+q\leq N-1}} \left(\frac{c}{2N}\right)^{p+q} \frac{(N-1)!}{p!q!(N-p-q-1)!} C_{1,N}Q_{1,2}(c) \cdots Q_{1,p+1}(c)Q_{p+2,1}(c) \cdots Q_{p+q+1,1}(c)h(x_1 - x_2) \cdots h(x_1 - x_{p+1})h(x_{p+2} - x_1) \cdots h(x_{p+q+1} - x_1)\Phi_N. \tag{4.4}$$

The operator $Q_{1,2}(c) \cdots Q_{p+q+1,1}(c)h(x_1 - x_2) \cdots h(x_{p+q+1} - x_1)$ in (4.4) does not act on variables $x_{p+q+2}, \dots, x_N$ therefore

$$C_{1,N}Q_{1,2}(c) \cdots Q_{p+q+1,1}(c)h(x_1 - x_2) \cdots h(x_{p+q+1} - x_1)\Phi_N = C_{1,p+q+1}Q_{1,2}(c) \cdots Q_{p+q+1,1}(c)h(x_1 - x_2) \cdots h(x_{p+q+1} - x_1)C_{p+q+1,N}\Phi_N. \tag{4.5}$$

Setting $\Phi_N = e^{tL_N}F_N$ where F_N is given by (2.1) we have by propagation of chaos (2.4)

$$C_{p+q+1,N}e^{tL_N}F_N \underset{N\to\infty}{\to} f(t;x_1) \cdots f(t;x_{p+q+1}). \tag{4.6}$$

It is clear from (1.5) that when $N \to \infty$, $Q_{i,j}(c) \to Q_{i,j}$ given by

$$(Q_{i,j}F)(x_1, \dots, x_n) = \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right) \times \int_0^\infty dt\, F(x_1, \dots, x_i - t, \dots, x_j + t, \dots, x_n). \tag{4.7}$$

To simplify notation we denote x_1 by x, x_2 by $x_1, \dots,$ x_{n+1} by x_n. Denote $C_{1,n}$ by C_n and $h(x - x_1) \cdots h(x_{p+q} - x)$ by $H(p, q)$. The argument above implies that when $N \to \infty$, expansion (4.4) for $\varphi_N = C_N R_N e^{tL_N}F_N$ converges term by term to the expansion

$$\varphi = \sum_{n=0}^{\infty} \left(\frac{c}{2}\right)^n \sum_{\substack{p+q=n \\ p,q\geq 0}} \frac{1}{p!q!} C_n Q_{1,2} \cdots Q_{1,p}Q_{p+1,1} \cdots Q_{n,1}H_{p,q}f(t;x)f(t;x_1) \cdots f(t;x_n). \tag{4.8}$$

Recall that $f(t; x)$ satisfies the Burgers Eq. (1.1) with the initial condition $f_0(x)$. As we have seen in Section 3, φ_N satisfies the one-dimensional heat equation with the initial condition $C_N R_N F_N$. Expansion (4.8) if it converges defines a nonlinear operator T on functions of t and x which has a homogeneous expansion $T = \sum_{n=0}^{\infty}(c/2)^n T_n$. Operator T does not depend on time so for brevity we supress t in calculations.

Denote by E_n the expansion operator

$$E_n : f(x) \to f(x) f(x_1) \cdots f(x_n). \tag{4.9}$$

Let $A_{p,q}$ be the linear operator from functions of $n = p + q + 1$ variables into functions of one variable given by

$$A_{p,q} = C_{p+q} Q_{1,2} \cdots Q_{1,p} Q_{p+1,1} \cdots Q_{p+q,1} \tag{4.10}$$

and let $H_{p,q}$ be the operator of multiplication by $H_{p,q}(x, x_1, \ldots, x_{p+q})$. Then the T operator takes form

$$T = \sum_{n=0}^{\infty} \left(\frac{c}{2}\right)^n \left(\sum_{p+q=n} \frac{1}{p!q!} A_{p,q} H_{p,q} \right) E_n \tag{4.11}$$

so the nonlinearity of T_n is due to E_n which has degree $n + 1$.

Our goal is to simplify the expression for $A_{p,q}$. If $\Phi(x, \ldots, x_n)$ is a function of $n + 1$ variables we have from (1.5) and (4.7)

$$\begin{aligned}(A_{p,q}\Phi)(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} \right) \cdots \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_n} \right) \\ \times \int_0^{\infty} \cdots \int_0^{\infty} dt_1 \cdots dt_n \qquad\qquad (4.12) \\ \Phi(x - t_1 - \cdots - t_p + t_{p+1} + \cdots + t_n, x + t_1, \ldots, x \\ + t_p, x - t_{p+1}, \ldots, x - t_n).\end{aligned}$$

We evaluate the integral (4.12) by induction on $n = p + q$. Set

$$\begin{aligned}\Psi(x, x_1, \ldots, x_n) = \int_0^{\infty} \cdots \int_0^{\infty} dt_1 \cdots dt_{n-1} \\ \Phi(x - t_1 - \cdots + t_{n-1}, x + t_1, \ldots, x - t_{n-1}, x_n).\end{aligned} \tag{4.13}$$

Then

$$(A_{p,q}\Phi)(x) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_{n-1} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} \right) \cdots \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_{n-1}} \right) \int_{-\infty}^{\infty} dx_n \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_n} \right) \times \int_0^{\infty} dt_n \Psi(x + t_n, x_1, \ldots, x_{n-1}, x_n - t_n). \tag{4.14}$$

Freezing the values of $x_1, \ldots, x_{n-1}$ we consider the inner integral in (4.14)

$$\int_{-\infty}^{\infty} dx_n \int_0^{\infty} dt_n \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_n} \right) \Psi(x + t_n, x_1, \ldots, x_{n-1}, x_n - t_n). \tag{4.15}$$

The change of variables $u = x + t_n, v = x_n - t_n$ in (4.15) yields

$$\begin{aligned} \iint_{u \geq x} du\, dv \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \Psi(u, x_1, \ldots, x_{n-1}, v) &= \int_x^{\infty} du \frac{\partial}{\partial u} \int_{-\infty}^{\infty} dv\, \Psi(u, x_1, \ldots, x_{n-1}, v) \\ &= -\int_{-\infty}^{\infty} dv\, \Psi(x, x_1, \ldots, x_{n-1}, v). \end{aligned} \tag{4.16}$$

Going back to the integral (4.12) we have

$$(A_{p,q}\Phi)(x) = -\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_{n-1} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_1} \right) \cdots \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x_{n-1}} \right) \int_0^{\infty} \cdots \int_0^{\infty} dt_1 \cdots dt_{n-1} \int_{-\infty}^{\infty} dx_n \Phi(x - t_1 - \cdots + t_{n-1}, x_1 + t_1, \ldots, x_{n-1} - t_{n-1}, x_n). \tag{4.17}$$

This integral is of the same type as (4.12) with n replaced by $n - 1$. Continuing the reduction we have after n steps

$$(A_{p,q}\Phi)(x) = (-1)^q \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n \Phi(x, x_1, \ldots, x_n). \tag{4.18}$$

Thus

$$(A_{p,q}H_{p,q}F)(x) = (-1)^q \int_{-\infty}^{x} dx_1 \cdots \int_{-\infty}^{x} dx_p \int_{x}^{\infty} dx_{p+1} \cdots \int_{x}^{\infty} dx_n F(x, x_1, \dots, x_n). \tag{4.19}$$

Since $F(x, x_1, \dots, x_n) = E_n f = f(x)f(x_1) \cdots f(x_n)$ the right-hand side of (4.19) is equal to

$$(-1)^q f(x) \left(\int_{-\infty}^{x} f(x)\, dx \right)^p \left(\int_{x}^{\infty} f(x)\, dx \right)^q. \tag{4.20}$$

Denoting for brevity $\int_{-\infty}^{x} f(x)\, dx$, $\int_{x}^{\infty} f(x)\, dx$ by $\int^{x} f$, $\int_{x} f$, respectively, we have from (4.8)

$$\begin{aligned}
\varphi(x) = (Tf)(x) &= \sum_{n=0}^{\infty} \left(\frac{c}{2}\right)^n \sum_{p+q=n} \frac{1}{p!q!} (-1)^q f(x) \left(\int^{x} f \right)^p \left(\int_{x} f \right)^q \\
&= f(x) \sum_{n=0}^{\infty} \left(\frac{c}{2}\right)^n \Big/ n! \sum_{p+q=n} \frac{n!}{p!q!} (-1)^q \left(\int^{x} f \right)^p \left(\int_{x} f \right)^q \\
&= f(x) \sum_{n=0}^{\infty} \left(\left(\frac{c}{2}\right)^n \Big/ n! \right) \left[\int^{x} f - \int_{x} f \right]^n \\
&= f(x) e^{(c/2)[\int^{x} f - \int_{x} f]} = c^{-1} \frac{\partial}{\partial x} e^{(c/2)[\int^{x} f - \int_{x} f]}.
\end{aligned} \tag{4.21}$$

Expansion (4.21) obviously converges which justifies taking the limit $\varphi = \lim_{N \to \infty} \varphi_N$.

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